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Stimulated Emission by External Sources in Quantum Field Theory

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A formalism is developed to study stimulated emission by external sources in relativistic quantum field theory as a generalization of an earlier work involving essentially noninteracting particles. A general expression is derived for transition amplitudes for the production of an arbitrary number of particles, as final products, by emission sources when there is initially an arbitrary number of particles before the intervention of the emission sources, thus stimulating the latter for further emissions. An application to quantum electrodynamics is then given in the presence of an external electromagnetic current with an initial background radiation of an arbitrary number of photons with unspecified momenta and spins leading to an electron-positron pair as final products.

1. INTRODUCTION

In a previous paper (Manoukian, 1986) a study was carried out for stimulated emission by external sources, essentially for noninteracting particles. The purpose of this paper is to generalize this work to the far more interesting case in relativistic *quantum field theory* (QFT) in the presence of external sources. The physical situation is the following. We have initially a number of particles, a so-called background system of particles; then an intervening external source is switched on, with the latter being stimulated by the background system of particles to emit a certain number of particles. Here we note at the outset that the *final* products of this complex process may, in general, be different from what the intervening source has initially emitted. For example, the initial particles produced by the external source as stimulated emissions may scatter off each other, creating other particles; or a virtual particle, not capable of propagating to macroscopic distances, may decay to other particles. The general analysis for the derivation of the

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transition amplitudes concerning such stimulated emission by the external sources in field theory is given in Section 2. An application of this work is then given to quantum electrodynamics (QED) in the presence of an external electromagnetic current in Section 3. Here we consider a situation where we have initially a background radiation of photons of arbitrary momenta and spins. The latter are then absorbed by the external current, thus stimulating it for further emission and finally having an electron-positron pair escaping to the detection region at macroscopic distances.

2. STIMULATED EMISSION BY EXTERNAL SOURCES IN QFT

For simplicity of the presentation we consider an interacting real pseudoscalar $\phi(x)$, with the latter coupled linearly to an external (c-number) source $K(x)$. The vacuum-to-vacuum transition amplitude is given by the well-known expression

$$\langle 0_+|0_-\rangle^K = \exp \sum_{n\geq 2} \frac{(i)^{n-1}}{n!} \int (dx_1)\cdots(dx_n)K(x_1)\cdots K(x_n)G_c(x_1,\ldots,x_n) \tag{1}$$

$$(dx) = dx^0\, dx^1\, dx^2\, dx^3 \tag{2}$$

where $G_c(x_1,\ldots,x_n)$ are the *connected* Green's functions (in the absence of external sources). In particular,

$$G_c(x,x') \equiv G(x-x') = \int_0^\infty d\mu^2\, \rho(\mu^2) \int \frac{(dk)}{(2\pi)^4} \frac{e^{ik(x-x')}}{k^2+\mu^2-i\varepsilon}, \qquad \varepsilon \to +0 \tag{3}$$

or

$$G(x-x') = i\int_0^\infty d\mu^2\, \rho(\mu^2) \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]}{2\sqrt{\mathbf{k}^2+\mu^2}} \times \exp[\pm i(\mathbf{k}^2+\mu^2)^{1/2}(x^0-x^{0\prime})] \tag{4}$$

where the minus sign is for $x^0 > x^{0'}$ and the plus sign is for $x^0 < x^{0'}$ in the last exponential in (4), and $\rho(\mu^2)$ denotes the two-point spectral function.

We write the external total source $K(x)$ as (Schwinger, 1970, 1973; Manoukian, 1986)

$$K(x) = K_1(x) + K_2(x) + K_3(x) \tag{5}$$

where $K_2(x)$, the intervening source, is switched on after the emission source $K_1(x)$ is switched off, and the detection source $K_3(x)$ is switched on after the source $K_2(x)$ is switched off. The source $K_1(x)$ acts as a preparatory stage for the emission of real (that is, on the mass shell) particles, which constitute the initial background of particles to the intervening source K_2. On the other hand, K_3 is the detection source, which detects and hence, "recognizes" the *final* products of real particles of the complex process of stimulated emission by the intervening source K_2. Since $K_1(x)$ and $K_3(x)$ deal with real particles on the mass shell, we write

$$K_1(x)=\int d^3\mathbf{x}'\, S_1(x-x')F(x'), \qquad x^{0'}\to-\infty \tag{6}$$

$$K_3(x)=\int d^3\mathbf{x}'\, S_3(x-x')F(x'), \qquad x^{0'}\to+\infty \tag{7}$$

where $F(x')$ is chosen so as to put the particles emitted and detected on the mass shell and may be conveniently chosen in the normalized form

$$\begin{aligned} F(x')&=\int\frac{(dp)}{(2\pi)^4}(2\pi)2|p^0|\,\delta(p^2+m^2)\,e^{ipx'} \\ &\equiv\int\frac{(dp)}{(2\pi)^4}F(p)\,e^{ipx} \end{aligned} \tag{8}$$

and $S_1(x)$, $S_3(x)$ are arbitrary:

$$S_j(x)=\int\frac{(dp)}{(2\pi)^4}S_j(p)\,e^{ipx}, \qquad S_j^*(p)=S_j(-p), \qquad j=1,3 \tag{9}$$

As will be seen [equations (34) and (35) below], $S_3^*(p)$, with $p^0=+(\mathbf{p}^2+m^2)^{1/2}$, is (proportional to) the *amplitude* that a particle of momentum $\mathbf{p}$ is at the detection region, and $S_1(p)$, with $p^0=+(\mathbf{p}^2+m^2)^{1/2}$, is (proportional to) the *amplitude* that a particle of momentum $\mathbf{p}$ is at the emission sight. The $x^{0'}\to\mp\infty$ limits in (6) and (7), respectively, ensure that the emission and detection regions are far from the interaction region where the actual physical process of stimulated emission takes place.

Upon defining

$$H_c(K_2;x_{n_2+1},\ldots,x_n)=\int\left(\prod_{j=1}^{n_2}(dx_j)\,K_2(x_j)\right)G_c(x_1,\ldots,x_n) \tag{10}$$

for $1\le n_2<n$,

$$H_c^{(n)}(K_2)=\int\left(\prod_{j=1}^{n}(dx_j)\,K_2(x_j)\right)G_c(x_1,\ldots,x_n) \tag{11}$$

we may rewrite the exponential factor in $\langle 0_+|0_-\rangle^K$ in (1) as

$$\begin{aligned} &\sum_{n\ge2}\frac{(i)^{n-1}}{n!}\sum_{(n_1+n_2+n_3=n)}\binom{n}{n_1\ n_2\ n_3}\int\left(\prod_{j=1}^{n-n_1}{}'\,(dx_j)\right) \\ &\quad\times K_1(x_1)\cdots K_1(x_{n_1})K_3(x_{n_1+1})\cdots K_3(x_{n_1+n_3})H_c(K_2;x_1,\ldots,x_{n-n_2}) \end{aligned} \tag{12}$$

Of particular interest are the bilinear terms in the sources K_1, K_3 in

(12). These are readily evaluated from (4) and (6)-(9) to be

$$2\,\mathrm{Re}\left(\frac{i}{2}K_jG_+K_j\right) = -Z\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{|S_j(k)|^2}{2k^0}, \qquad j=1,3 \tag{13}$$

$$iK_1G_+K_3 = Z\int \frac{d^3\mathbf{k}}{(2\pi)^3 2k^0} iS_3^*(k)iS_1(k) \tag{14}$$

with $k^0 = +(\mathbf{k}^2+m^2)^{1/2}$, where we have used the spectral form

$$\rho(\mu^2) = Z\delta(\mu^2-m^2) + \text{continuum} \tag{15}$$

in conjunction with (8).

Similarly, if we define the free propagator G_+^0 (expressed in terms of the physical mass)

$$G_+^0(x-y) = \int \frac{(dp)}{(2\pi)^4}\frac{e^{ip(x-y)}}{p^2+m^2-i\varepsilon}, \qquad \varepsilon \to +0 \tag{16}$$

we obtain

$$\int (dx)\; K_3(x)G_+^0(x-y) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k^0} iS_3^*(k)\; e^{-iky} \tag{17}$$

$$\int (dx)\; G_+^0(y-x)K_1(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k^0} iS_1(k)\; e^{iky} \tag{18}$$

with $k^0 = +(k^2+m^2)^{1/2}$. Finally, we set, for $j>2$,

$$H_A(K_2; x_1, \ldots, x_j) = (-\Box_1^2+m^2)\cdots(-\Box_j^2+m^2)H_c(K_2; x_1, \ldots, x_j) \tag{19}$$

$$\begin{aligned} H_A(K_2; x_1, \ldots, x_j) = \int & (dx_1)\cdots(dx_j)\; H_A(K_2; k_1, \ldots, k_j) \\ & \times \exp[i(k_1x_1+\cdots+k_jx_j)] \end{aligned} \tag{20}$$

and define

$$H_A(K_2; k_1, -k_2) = Z\delta^3(\mathbf{k}_1-\mathbf{k}_2)d^3\mathbf{k}_2 \tag{21}$$

to rewrite (1) from (12) and (17)-(21) as

$$\langle 0_+|0_-\rangle^K = \exp(\tfrac{1}{2}iK_3G_+K_3)\exp(F[S_3, S_1])\; A\, \exp(\tfrac{1}{2}iK_1G_+K_1) \tag{22}$$

where, from (11),

$$A = \exp\left(\sum_{n\geq 2}\frac{(i)^{n-1}}{n!}H_c^{(n)}(K_2)\right) \tag{23}$$

$$F[S_3, S_1] = \sum_{n\geq 2}\frac{(i)^{n-1}}{n!}\sum_{(n_1+n_2+n_3=n)}{}' \binom{n}{n_1\;\; n_2\;\; n_3}$$

$$\times \int \left(\prod_{j=1}^{n_3} \frac{d^3\mathbf{k}_j}{(2\pi)^3 2k_j^0} iS_3^*(k_j) \right)$$
$$\times H_A(K_2; k_1, \ldots, k_{n_3}, -k_{n_3+1}, \ldots, -k_{n-n_2})$$
$$\times \left(\prod_{j=n_3+1}^{n-n_2} \frac{d^3\mathbf{k}_j}{(2\pi)^3 2k_j^0} iS_1(k_j) \right) \tag{24}$$

where $\sum'_{(n_1+n_2+n_3=n)}$ stands for $\sum_{(n_1+n_2+n_3=n)}$ with the following terms omitted: $(n_3=n, n_1=0, n_2=0)$, $(n_1=n, n_3=0, n_2=0)$. The $(n_3=n=2, n_1=0, n_2=0)$ term is included in the first exponential term on the right-hand side of (22). On the other hand, for $(n_3=n>2, n_1=0, n_2=0)$, we may use the fact that $H_A(K_2; k_1, \ldots, k_n) \equiv H_A(0; k_1, \ldots, k_n)$ is *independent* of the external source K_2; hence, we use translational invariance to write

$$H_A(0; k_1, \ldots, k_n) \equiv \delta(k_1 + \cdots + k_n) \tilde{H}_A(k_1, \ldots, k_n) \tag{25}$$

$$\delta(k_1^0 + \cdots + k_n^0) = 0 \tag{26}$$

for $k^0 = (\mathbf{k}^2 + m^2)^{1/2}$, $m > 0$. [For $m=0$, the $(n_3=n>2, n_1=0, n_2=0)$ will again not contribute if $\tilde{H}_A(k_1, \ldots, k_n) = 0$ when any one or more of the $k_j = 0$, as is the case in light-light scattering in QED due to gauge invariance.] A similar analysis holds for the $(n_1=n, n_3=0, n_2=0)$ terms.

In a convenient discrete momentum notation (Schwinger, 1970, 1973; Manoukian, 1984, 1986) we rewrite $F[K_3, K_1]$ in (24) as

$$F[S_3, S_1] = \sum_{n \geq 2} \frac{(i)^{n-1}}{n!} \sum'_{n_1+n_2+n_3=n} \begin{pmatrix} & n & \\ n_1 & n_2 & n_3 \end{pmatrix} \sum_{\mathbf{k}_1, \ldots, \mathbf{k}_{n-n_2}}$$
$$\times iS^*_{3\mathbf{k}_1} \cdots iS^*_{3\mathbf{k}_{n_3}} \hat{H}_A(K_2; k_1, \ldots, k_{n_3}, -k_{n_3+1}, \ldots, -k_{n-n_2})$$
$$\times iS_{1\mathbf{k}_{n_3+1}} \cdots iS_{1\mathbf{k}_{n-n_2}} \tag{27}$$

where

$$S^*_{3\mathbf{k}} = \left[\frac{d^3\mathbf{k}}{(2\pi)^3 2k^0} \right]^{1/2} S_3^*(k) \tag{28}$$

$$S_{1\mathbf{k}} = \left[\frac{d^3\mathbf{k}}{(2\pi)^3 2k^0} \right]^{1/2} S_1(k) \tag{29}$$

$$\hat{H}_A(K_2; k_1, \ldots, k_{n_3}, -k_{n_3+1}, \ldots, -k_{n-n_2})$$
$$= \left\{ \prod_{j=1}^{n-n_2} \left[\frac{d^3\mathbf{k}_j}{(2\pi)^3 2k_j^0} \right]^{1/2} \right\} H_A(K_2, k_1, \ldots, k_{n_3}, -k_{n_3+1}, \ldots, -k_{n-n_2}) \tag{30}$$

Now we are ready to compare the expression in (22) with a unitarity expansion (Schwinger, 1970, Manoukian, 1986):

$$\langle 0_+|0_-\rangle^K = \sum_{N=0}^{\infty} \sum_{N_{\mathbf{k}_1}+N_{\mathbf{k}_2}+\cdots=N} \sum_{M=0}^{\infty} \sum_{M_{\mathbf{k}_1}+M_{\mathbf{k}_2}+\cdots=M} \times\langle 0_+|N; N_{\mathbf{k}_1}, N_{\mathbf{k}_2}, \ldots +\rangle^{K_3}$$
$$\times\langle N; N_{\mathbf{k}_1}, N_{\mathbf{k}_2}, \ldots +|M; M_{\mathbf{k}_1}, M_{\mathbf{k}_2}, \ldots -\rangle^{K_2}$$
$$\times\langle M; M_{\mathbf{k}_1}, M_{\mathbf{k}_2}, \ldots -|0_-\rangle^{K_1} \tag{31}$$

where

$$\langle N; N_{\mathbf{k}_1}, N_{\mathbf{k}_2}, \ldots +|M; M_{\mathbf{k}_1}, M_{\mathbf{k}_2}, \ldots -\rangle^{K_2}$$
$$\equiv\langle N; N_{\mathbf{k}_1}, N_{\mathbf{k}_2}, \ldots \text{out}|M; M_{\mathbf{k}_1}, M_{\mathbf{k}_2}, \ldots \text{in}\rangle^{K_2} \tag{32}$$

is the object of interest and represents the *transition amplitude* for stimulated emission of N particles as *final* products with various momenta, when there are initially M particles with various momenta, by the intervening source K_2, and the former obviously depends on K_2.

By carrying out a Taylor expansion in S_3^* and S_1 of $\exp(F[S_3, S_1])$ with the latter as defined in (27), we obtain

$$\exp(F[S_3, S_1]) = \sum_{N=0}^{\infty} \sum_{N_{\mathbf{k}_1}+N_{\mathbf{k}_2}+\cdots=N} \sum_{M=0}^{\infty} \sum_{M_{\mathbf{k}_1}+M_{\mathbf{k}_2}+\cdots=M}$$
$$\times\frac{(S_{3\mathbf{k}_1}^*)^{N_{\mathbf{k}_1}}}{N_{\mathbf{k}_1}!}\frac{(S_{3\mathbf{k}_2}^*)^{N_{\mathbf{k}_2}}}{N_{\mathbf{k}_2}!}\cdots\frac{(S_{1\mathbf{k}_1})^{M_{\mathbf{k}_1}}}{M_{\mathbf{k}_1}!}\frac{(S_{1\mathbf{k}_2})^{M_{\mathbf{k}_2}}}{M_{\mathbf{k}_2}!}\cdots$$
$$\times\left(\frac{\delta}{\delta T_{3\mathbf{k}_1}^*}\right)^{N_{\mathbf{k}_1}}\left(\frac{\delta}{\delta T_{3\mathbf{k}_2}^*}\right)^{N_{\mathbf{k}_2}}\cdots\left(\frac{\delta}{\delta T_{1\mathbf{k}_1}}\right)^{M_{\mathbf{k}_1}}\left(\frac{\delta}{\delta T_{1\mathbf{k}_2}}\right)^{M_{\mathbf{k}_2}}\cdots$$
$$\times\exp(F[T_3, T_1])\bigg|_{T_3=0, T_1=0} \tag{33}$$

Hence, from (22), (27), (31), and (33) we obtain

$$\langle 0_+|N; N_{\mathbf{k}_1}, N_{\mathbf{k}_2}, \ldots +\rangle^{K_3}$$
$$=\exp\left(\frac{i}{2}K_2G_+K_3\right)\frac{(Z^{1/2}iS_{3\mathbf{k}_1}^*)^{N_{\mathbf{k}_1}}}{(N_{\mathbf{k}_1}!)^{1/2}}\frac{(Z^{1/2}iS_{3\mathbf{k}_2}^*)^{N_{\mathbf{k}_2}}}{(N_{\mathbf{k}_2}!)^{1/2}}\cdots \tag{34}$$

$$\langle M; M_{\mathbf{k}_1}, M_{\mathbf{k}_2}, \ldots -|0_-\rangle^{K_1}$$
$$=\exp\left(\frac{i}{2}K_1G_+K_1\right)\frac{(Z^{1/2}iS_{1\mathbf{k}_1})^{M_{\mathbf{k}_1}}}{(M_{\mathbf{k}_1}!)^{1/2}}\frac{(Z^{1/2}iS_{1\mathbf{k}_2})^{M_{\mathbf{k}_2}}}{(M_{\mathbf{k}_2}!)^{1/2}}\cdots \tag{35}$$

$$\langle N; N_{\mathbf{k}_1}, N_{\mathbf{k}_2}, \ldots \text{out}|M; M_{\mathbf{k}_1}, M_{\mathbf{k}_2}, \ldots \text{in}\rangle^{K_2}$$
$$=\frac{(Z)^{-(N+M)/2}}{(N_{\mathbf{k}_1}!\,N_{\mathbf{k}_2}!\cdots M_{\mathbf{k}_1}!\,M_{\mathbf{k}_2}!\cdots)^{1/2}}A\left(-i\frac{\delta}{\delta T_{3\mathbf{k}_1}^*}\right)^{N_{\mathbf{k}_1}}\left(-i\frac{\delta}{\delta T_{3\mathbf{k}_2}^*}\right)^{N_{\mathbf{k}_2}}\cdots$$

$$\times\left(-i\frac{\delta}{\delta T_{1\mathbf{k}_1}}\right)^{M_{\mathbf{k}_1}}\left(-i\frac{\delta}{\delta T_{1\mathbf{k}_2}}\right)^{M_{\mathbf{k}_2}}\cdots \exp(F[T_3, T_1])\bigg|_{T_3=0, T_1=0} \tag{36}$$

where A is defined in (23) and $F[T_3, T_1]$ is defined in (27). The wavefunction renormalization function contribution $(Z)^{-(N+M)/2}$ in (36) arises as a consequence of the *completeness relations* that have to be satisfied:

$$\sum_{N=0}^{\infty}\sum_{N_{\mathbf{k}_1}+N_{\mathbf{k}_2}+\cdots=N}|\langle 0_+|N;\,N_{\mathbf{k}_1}, N_{\mathbf{k}_2},\ldots +\rangle^{K_3}|^2=1 \tag{37}$$

$$\sum_{M=0}^{\infty}\sum_{M_{\mathbf{k}_1}+M_{\mathbf{k}_2}+\cdots=M}|\langle M;\,M_{\mathbf{k}_1}, M_{\mathbf{k}_2},\ldots -|0_-\rangle^{K_1}|^2=1 \tag{38}$$

as are readily verified from (13), (34), and (35).

The expression in (36) gives the final general form for the transition amplitudes in question for stimulated emission and includes connected and disconnected processes. In particular, A denotes the amplitude that an arbitrary number of particles is produced by the source K_2 and after their interactions the final products are absorbed back by the source.

3. APPLICATION

We apply our formalism to QED, to lowest order in the fine structure constant, in the presence of an external conserved electromagnetic current $J^2_\mu(x)$: $\partial^\mu J^2_\mu(x)=0$, $Q^\mu J^2_\mu(Q)=0$. We consider the situation where initially we have a background radiation of photons of arbitrary momenta and spins before the intervening current J^2_μ is switched on and finally have an electron–positron pair escaping to a detection region, with the momenta and spins of the latter particles unspecified.

We modify the QED Lagrangian density $\mathscr{L}_{\mathrm{QED}}$ by adding source terms:

$$\mathscr{L}_{\mathrm{QED}}(x)+\bar\eta(x)\psi(x)+\bar\psi(x)\eta(x)+A^\mu(x)J_\mu(x) \tag{39}$$

where η and $\bar\eta$ are Fermi sources, which are eventually set to *zero*, and in this limit we also set $J_\mu(x)\to J^2_\mu(x)$ with $\partial^\mu J^2_\mu(x)=0$. Then the amplitudes for finding an electron and a positron in a detection region and a photon in an emission region are, respectively, proportional to (Manoukian, 1986)

$$(2m\,d\omega_{\mathbf{p}})^{1/2}\bar\eta(p)u(p,\sigma)=\eta^*_{\mathbf{p}\sigma-} \tag{40}$$

$$(2m\,d\omega_{\mathbf{p}})^{1/2}\bar v(p,\sigma)\eta(-p)=\eta^*_{\mathbf{p}\sigma+} \tag{41}$$

$$(d\omega_{\mathbf{q}})^{1/2}e^\mu(q,\lambda)J^2_\mu(q)=J^2_{\mathbf{q}\lambda} \tag{42}$$

where

$$d\omega_{\mathbf{p}}=\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2p^0} \tag{43}$$

$$p^0 = +(\mathbf{p}^2 + m^2)^{1/2}, \qquad q^0 = |\mathbf{q}| \tag{44}$$

$$\sum_{\sigma} u(p, \sigma)\bar{u}(p, \sigma) = (-\gamma p + m)/2m \tag{45}$$

$$\sum_{\sigma} v(p, \sigma)\bar{v}(p, \sigma) = -(\gamma p + m)/2m \tag{46}$$

and (Schwinger, 1970)

$$g^{\mu\nu} = \frac{q^\mu \bar{q}^\nu + \bar{q}^\mu q^\nu}{(q\bar{q})} + \sum_{\lambda=1,2} e^\mu(q, \lambda) e^\nu(q, \lambda)^* \tag{47}$$

$$q^\mu = (|\mathbf{q}|, \mathbf{q}), \qquad \bar{q}^\mu = (|\mathbf{q}|, -\mathbf{q}) \tag{48}$$

The amplitude in question is then readily obtained from (36) to be, where all the initial photons are absorbed by the external current J^2_μ,

$$\begin{aligned} \mathscr{A} &\equiv \langle 2; (1_{\mathbf{p}_1}, \sigma, -), (1_{\mathbf{p}_2}, \sigma', +)\text{out}|N; N_{\mathbf{q}_1\lambda_1}, N_{\mathbf{q}_2\lambda_2}, \ldots \text{in}\rangle^{J_2} \\ &= ie(2m\, d\omega_{\mathbf{p}_1})^{1/2}(2m\, d\omega_{\mathbf{p}_2})^{1/2} \frac{\bar{u}(p_1, \sigma)\gamma^\mu v(p_2, \sigma')}{(p_1 - p_2)^2} J^2_\mu(p_1 - p_2) \\ &\quad \times \frac{(J^2_{\mathbf{q}_1\lambda_1})^{N_{\mathbf{q}_1\lambda_1}}}{(N_{\mathbf{q}_1\lambda_1}!)^{1/2}} \frac{(J^2_{\mathbf{q}_2\lambda_2})^{N_{\mathbf{q}_2\lambda_2}}}{(N_{\mathbf{q}_2\lambda_2}!)^{1/2}} \cdots \exp\left(\frac{i}{2} J D_+ J\right) \end{aligned} \tag{49}$$

Let

$$Q = p_1 + p_2 - N_{\mathbf{q}_1\lambda_1} q_1 - N_{\mathbf{q}_2\lambda_2} q_2 - \cdots$$

Then the transition probability of the process in question may be written as

$$\begin{aligned} \sum_{\substack{\mathbf{p}_1, \mathbf{p}_2 \\ \sigma, \sigma'}} (2\pi)^4 \sum_{N=0} \sum_{N_{\mathbf{q}_1\lambda_1} + N_{\mathbf{q}_2\lambda_2} + \cdots = N} \delta(p_1 + p_2 - N_{\mathbf{q}_1\lambda_1} q_1 \\ - N_{\mathbf{q}_2\lambda_2} q_2 - \cdots - Q)|\mathscr{A}|^2 \end{aligned} \tag{50}$$

or, upon using the integral representation

$$\begin{aligned} &\delta(p_1 + p_2 - N_{\mathbf{q}_1\lambda_1} q_1 - N_{\mathbf{q}_2\lambda_2} q_2 - \cdots - Q) \\ &= \frac{1}{(2\pi)^4} \int (dx) \exp i[p_1 + p_2 - N_{\mathbf{q}_1\lambda_1} q_1 - N_{\mathbf{q}_2\lambda_2} q_2 - \cdots - Q]x \end{aligned} \tag{51}$$

and the identity (Manoukian, 1984, 1986)

$$\begin{aligned} &\sum_{N_{\mathbf{q}_1\lambda_1} + N_{\mathbf{q}_2\lambda_2} + \cdots = N} \frac{(|J_{\mathbf{q}_1\lambda_1}|^2 e^{-iq_1 x})^{N_{\mathbf{q}_1\lambda_1}}}{N_{\mathbf{q}_1\lambda_1}!} \frac{(|J_{\mathbf{q}_2\lambda_2}|^2 e^{-iq_2 x})^{N_{\mathbf{q}_2\lambda_2}}}{N_{\mathbf{q}_2\lambda_2}!} \cdots \\ &= \left(\sum_{\mathbf{q}, \lambda} |J_{\mathbf{q}\lambda}|^2 e^{-iqx}\right)^N \Big/ N! \end{aligned} \tag{52}$$

and evaluating in the process the photon spectral function to lowest order (Schwinger, 1973; Nishijima, 1969), we obtain from (50) for the transition probability in question

$$\frac{\alpha}{3\pi}\int (dx)\,[\exp(-iQx)]\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int_{4m^2}^{\infty}\frac{dM^2}{M^2}\frac{1}{2k^0}\left(1+\frac{2m^2}{M^2}\right)$$
$$\times\left(1-\frac{4m^2}{M^2}\right)^{1/2}[J_2^{\mu}(k)^*J_{\mu}^2(k)]\exp(ikx)\exp\left(\int\frac{d^3\mathbf{q}}{(2\pi)^3 2|\mathbf{q}|}\right.$$
$$\left.\times[J_2^{\nu}(q)^*J_{\nu}^2(q)]\{\exp[i(\mathbf{q}\cdot\mathbf{x}-|\mathbf{q}|x^0)]-1\}\right) \tag{53}$$

where $k^0=+(\mathbf{k}^2+M^2)^{1/2}$, that is, $k^2=-M^2$, $q^0=|\mathbf{q}|$.

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